§ 13.1 Vector Valued Functions
(Newton's Theory & Kepler's Laws)

$$\vec{r}(t) = x(t)\,\underset{\sim}{i} + y(t)\,\underset{\sim}{j} + z(t)\,\underset{\sim}{k}$$

- Eg: Needed for vector version of Newton's Force Law

$$\vec{F} = m\vec{a}$$

- Main Point: $\vec{r}(t)$ gives position

$$\vec{v} = \vec{r}\,'(t) = x'(t)\,\underset{\sim}{i} + y'(t)\,\underset{\sim}{j} + z'(t)\,\underset{\sim}{k}$$

$$\vec{a} = \vec{r}\,''(t) = x''(t)\,\underset{\sim}{i} + y''(t)\,\underset{\sim}{j} + z''(t)\,\underset{\sim}{k}$$

$\vec{v}$ = velocity, $\vec{a}$ = acceleration (vectors!)

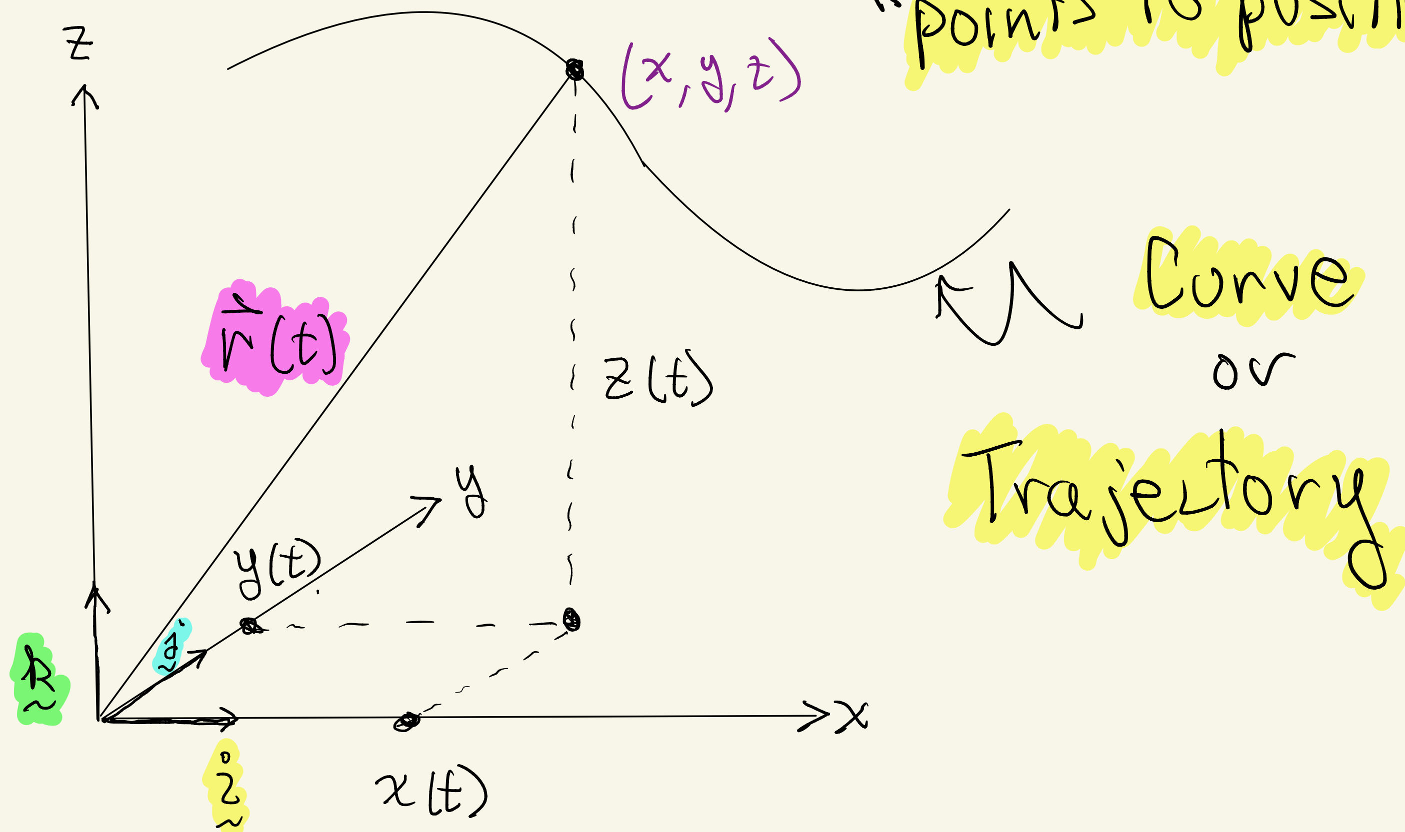
- To talk about derivatives we have to talk about limits

- Picture:

$\vec{r}(t) = \overrightarrow{(x(t), y(t), z(t))}$

= position vector

"points to position"



$\underset{\sim}{i} = \overrightarrow{(1,0,0)}$, $\underset{\sim}{j} = \overrightarrow{(0,1,0)}$, $\underset{\sim}{k} = \overrightarrow{(0,0,1)}$

Thus: $\vec{r}(t) = \overrightarrow{(x(t), y(t), z(t))}$

or: $\vec{r}(t) = x(t)\overrightarrow{(1,0,0)} + y(t)\overrightarrow{(0,1,0)} + z(t)\overrightarrow{(0,0,1)}$

$= x(t)\,\underset{\sim}{i} + y(t)\,\underset{\sim}{j} + z(t)\,\underset{\sim}{k}$

- What we want to see:

① $\vec{v}(t) = \vec{r}\,'(t)$ points tangent to curve

$$\vec{v}(t) = x'(t)\,\underset{\sim}{i} + y'(t)\,\underset{\sim}{j} + z'(t)\,\underset{\sim}{k}$$

② $\|\vec{v}(t)\|$ = speed $\frac{ds}{dt}$ at time t

$$\|\vec{v}(t)\| = \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}$$

③ $\vec{a}(t) = \vec{v}\,'(t) = \vec{r}\,''(t)$

$$\vec{a}(t) = x''(t)\,\underset{\sim}{i} + y''(t)\,\underset{\sim}{j} + z''(t)\,\underset{\sim}{k}$$

This is what you put into

$$\vec{F} = m\vec{a}$$

- **Basic Example**: Uniform Circular Motion in the Plane

$$\vec{r}(t) = \underbrace{\cos t}_{x(t)}\, \underset{\sim}{i} + \underbrace{\sin t}_{y(t)}\, \underset{\sim}{j}$$

Problem: Show $\vec{a}(t) \perp \vec{v}(t)$

Soln: $\vec{v}(t) = \vec{r}\,'(t) = -\sin t\, \underset{\sim}{i} + \cos t\, \underset{\sim}{j}$

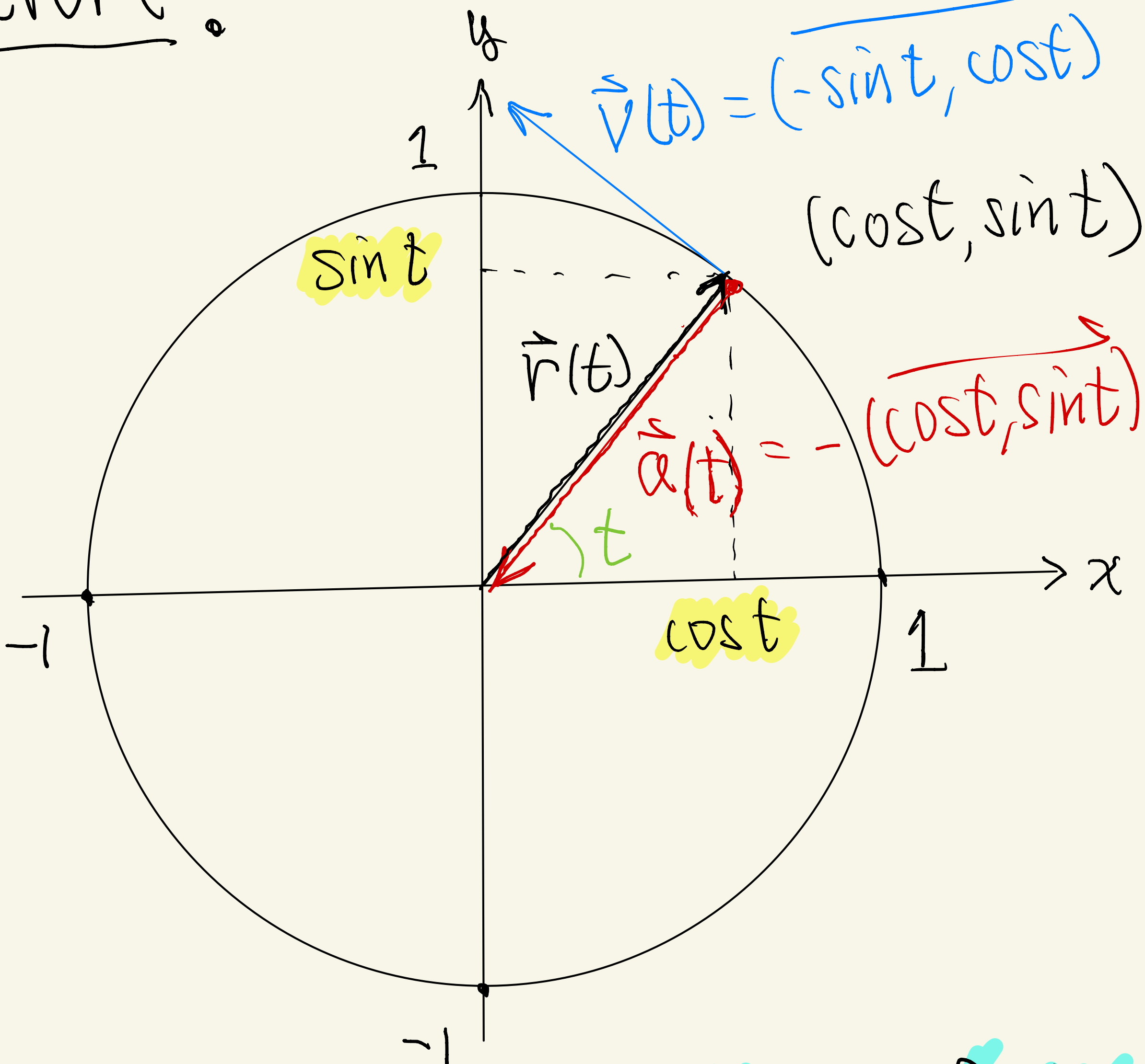
$\vec{a}(t) = \vec{r}\,''(t) = -\cos t\, \underset{\sim}{i} - \sin(t)\, \underset{\sim}{j}$

- $\vec{a}(t) = -\vec{r}(t)$
- $\vec{a}(t) \cdot \vec{v}(t) = \overrightarrow{(-\cos t, -\sin t)} \cdot \overrightarrow{(-\sin t, \cos t)}$

$= \cos t \sin t - \sin t \cos t = 0$

$\vec{a} \perp \vec{v}$!!

Picture:



• Conclude: If a force field $\vec{F}$ creates uniform circular motion,

$$\vec{F} = m\vec{a} = m\vec{r}''(t)$$

then the force is $\perp$ velocity...

... and points opposite the position

$$\vec{a}(t) = -\vec{r}(t)$$

• <u>In particular</u>: The earth moves in (approximate) circular orbit around the sun...

so this must be (approximately) true for the earth —

This is what gave Newton the idea that, if he defined $\vec{F} = m\vec{a}$, then he could explain why the earth was moving around the sun — his new theory (1687) was that the sun was "pulling" on the earth with a "gravitational force".

- Newton Gravitational Force:

$$\vec{F} = m\vec{a} = -\underbrace{G\frac{M_s M_e}{r^2}}_{\text{magnitude of force}}\frac{\vec{r}}{r}$$

Force points opposite to position vector

$r = \|\vec{r}\|$

G = Newtons constant

M_s = mass of sun

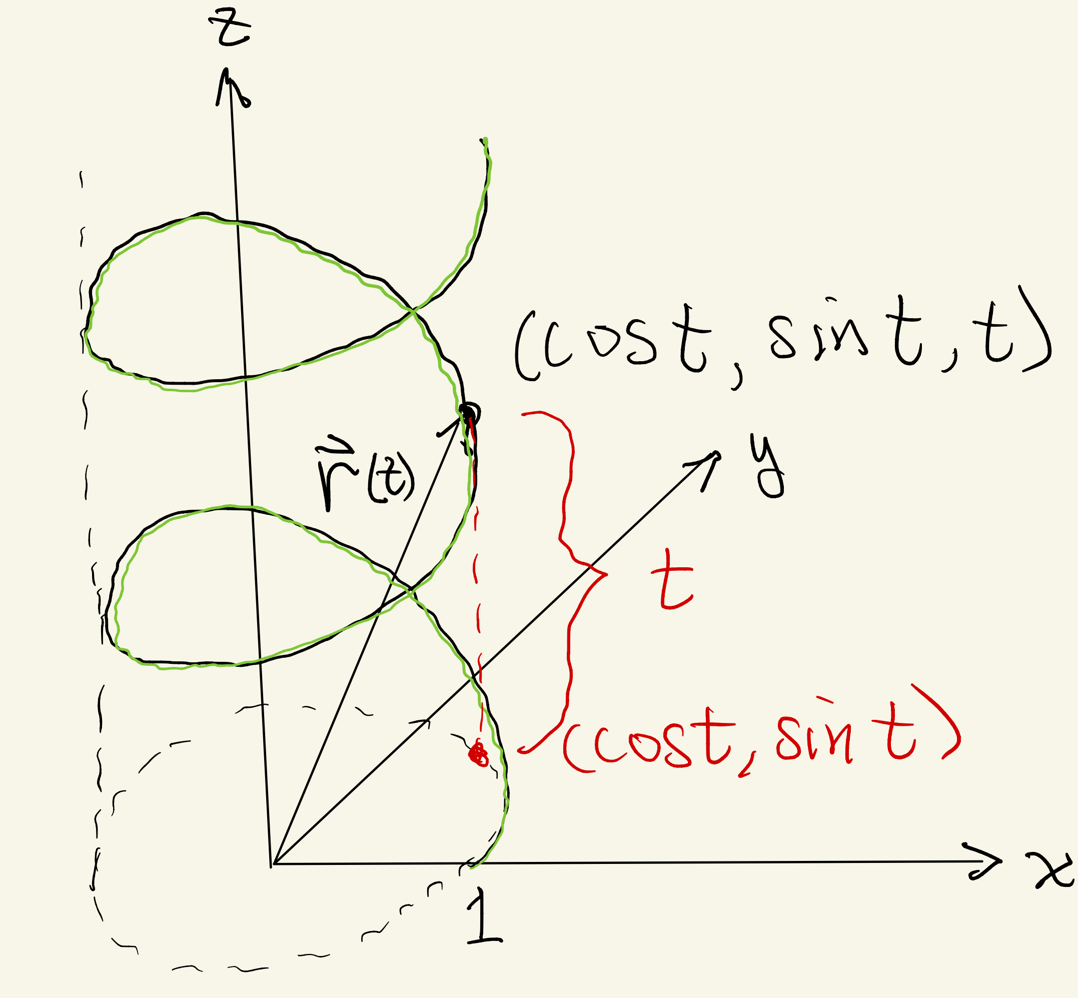
M_e = mass of earth

We're just shown the theory works for uniform circular motion - Newton then showed it works for elliptical orbits & Keplers Laws

Another Basic Example: Helix

$$\vec{r}(t) = \underbrace{\cos t}_{x(t)}\, \underset{\sim}{i} + \underbrace{\sin t}_{y(t)}\, \underset{\sim}{j} + \underbrace{t}_{z(t)}\, \underset{\sim}{k}$$

Graph:



$$\cos t\, \underset{\sim}{i} + \sin t\, \underset{\sim}{j} = \text{point on unit circle}$$

$$z\, \underset{\sim}{j} = \text{height}$$

Problem: $\vec{r}(t) = \cos t\, \underset{\sim}{i} + \sin t\, \underset{\sim}{j} + t\, \underset{\sim}{k}$

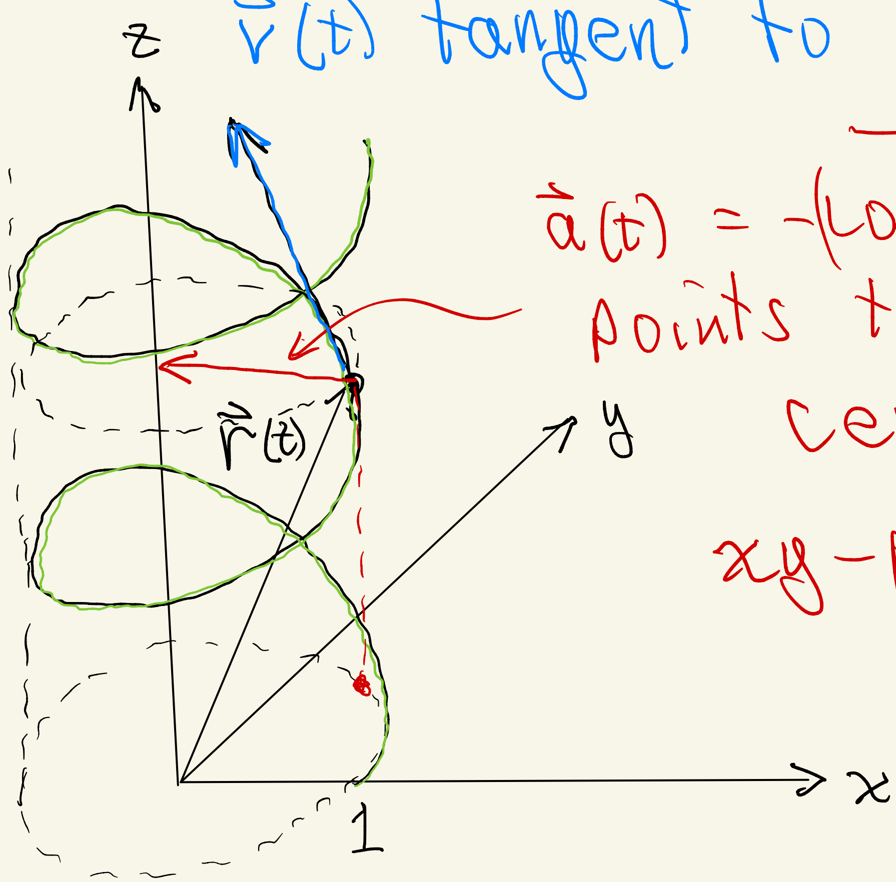
Find $\vec{v}$ and $\vec{a}$

Solution:

$$\vec{v} = \vec{r}\,'(t) = -\sin t\, \underset{\sim}{i} + \cos t\, \underset{\sim}{j} + \underset{\sim}{k}$$

$$\vec{a} = \vec{r}\,''(t) = -\cos t\, \underset{\sim}{i} - \sin t\, \underset{\sim}{j}$$

Eg: "zero acceleration in z-component"



$\vec{a}(t) = -\overrightarrow{(\cos t, \sin t)}$ points toward center in xy-plane

☒ **General Theorem**: If

$$\vec{r}(t) = x(t)\,\underset{\sim}{i} + y(t)\,\underset{\sim}{j} + z(t)\,\underset{\sim}{k}$$

then $\vec{v}(t)$ points tangent to the curve, and

$$\|\vec{v}(t)\| = \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2} = \frac{ds}{dt}$$

is the speed.

Proof: Start with the definition:

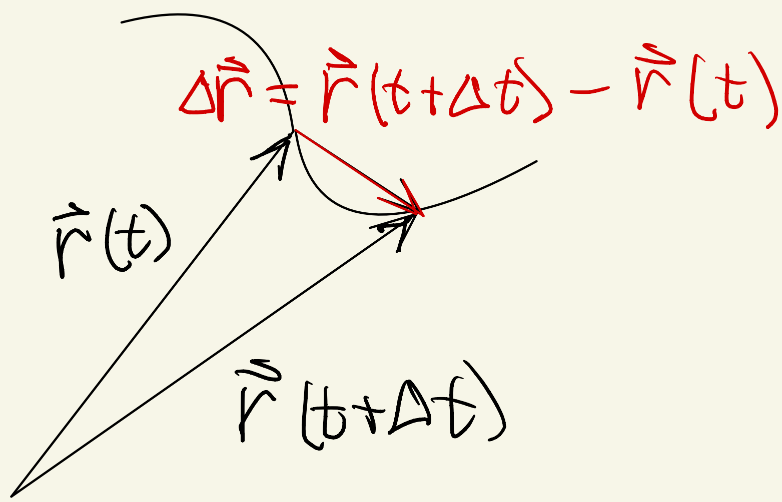
$$\vec{r}\,'(t) = \lim_{\Delta t \to 0}\left[\frac{\overbrace{\vec{r}(t+\Delta t) - \vec{r}(t)}^{\text{vector}}}{\Delta t} = \frac{\Delta\vec{r}}{\Delta t}\right]$$

vector

scalar

$$\vec{r}\,'(t) = \lim_{\Delta t \to 0} \left[\frac{\vec{r}(t+\Delta t) - \vec{r}(t)}{\Delta t} = \frac{\Delta \vec{r}}{\Delta t} \right]$$

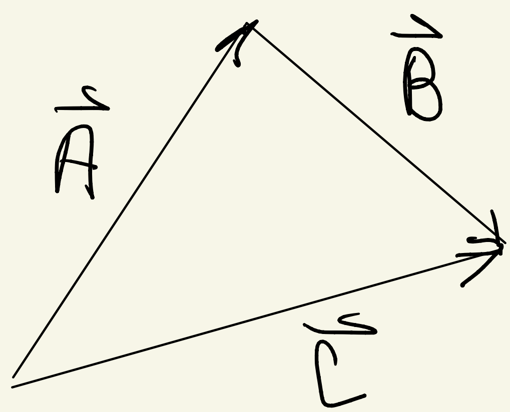
Geometrically it is clear that in the limit $\Delta t \to 0$, $\vec{r}\,'(t)$ will be tangent to the curve at $\vec{r}(t)$.



Recall vector addition

$$\vec{A} + \vec{B} = \vec{C}$$

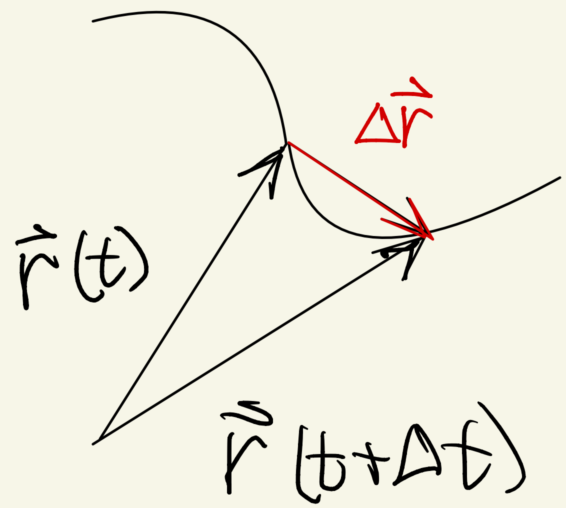
$$\vec{C} - \vec{A} = \vec{B}$$



- To get the speed — Note that

$\Delta s \approx$ length of $\Delta \vec{r} = \|\Delta \vec{r}\|$

So —
$$\frac{\Delta s}{\Delta t} \approx \frac{\|\Delta \vec{r}\|}{\Delta t} = \text{speed}$$



Thus —

$$\text{speed} = \frac{\text{dist}}{\text{time}} = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t} =$$

$$= \lim_{\Delta t \to 0} \frac{\|\Delta \vec{r}\|}{\Delta t} = \left\| \lim_{\Delta t \to 0} \frac{\Delta \vec{r}(t)}{\Delta t} \right\|$$

$$= \|\vec{r}\,'(t)\|$$

- A technical point –

Recall we defined the velocity as $\vec{v}(t) = x'(t)\,\underset{\sim}{i} + y'(t)\,\underset{\sim}{j} + z'(t)\,\underset{\sim}{k}$

Q: Is it true that also

$$\vec{v}(t) = \lim_{\Delta t \to 0} \frac{\Delta \vec{r}}{\Delta t} \; ?$$

Ans: Yes! Here's a proof...

We need to show

$$\lim_{\Delta t \to 0} \frac{\Delta \vec{r}}{\Delta t} = x'(t)\,\underset{\sim}{i} + y'(t)\,\underset{\sim}{j} + z'(t)\,\underset{\sim}{k}$$

To see this write...

$$\frac{\Delta\vec{r}}{\Delta t} = \frac{\vec{r}(t+\Delta t) - \vec{r}(t)}{\Delta t}$$

$$= \frac{\underset{\sim}{x(t+\Delta t)\,\underset{\sim}{i}} + \underset{\sim}{y(t+\Delta t)\,\underset{\sim}{j}} + \underset{\sim}{z(t+\Delta t)\,\underset{\sim}{k}} - \underset{\sim}{x(t)\,\underset{\sim}{i}} - \underset{\sim}{y(t)\,\underset{\sim}{j}} - \underset{\sim}{z(t)\,\underset{\sim}{k}}}{\Delta t}$$

$$= \frac{\left[x(t+\Delta t) - x(t)\right]\underset{\sim}{i} + \left[y(t+\Delta t) - y(t)\right]\underset{\sim}{j} + \left[z(t+\Delta t) - z(t)\right]\underset{\sim}{k}}{\Delta t}$$

$$= \underbrace{\frac{x(t+\Delta t) - x(t)}{\Delta t}}_{x'(t)}\underset{\sim}{i} + \underbrace{\frac{y(t+\Delta t) - y(t)}{\Delta t}}_{y'(t)}\underset{\sim}{j} + \underbrace{\frac{z(t+\Delta t) - z(t)}{\Delta t}}_{z'(t)}\underset{\sim}{k}$$

$$= x'(t)\,\underset{\sim}{i} + y'(t)\,\underset{\sim}{j} + z'(t)\,\underset{\sim}{k} \qquad \checkmark$$

- **Example:** A bead moves along a helix with position vector

$$\vec{r}(t) = 2(\cos 3t\, \underset{\sim}{i} + \sin 3t\, \underset{\sim}{j}) + 5t\, \underset{\sim}{k}$$

(assume length in meters m and time in seconds s)

(a) At what speed does it move?

Soln: $\dfrac{ds}{dt} = \left\| \dfrac{d\vec{r}}{dt} \right\| = \|\vec{v}(t)\|$

$\vec{v}(t) = \vec{r}\,'(t) = 6\sin 3t\, \underset{\sim}{i} + 6\cos 3t\, \underset{\sim}{j} + 5\underset{\sim}{k}$

$\|\vec{v}(t)\| = \sqrt{(6\sin 3t)^2 + (6\cos 3t)^2 + 5^2}$

$= \sqrt{36\sin^2 3t + 36\cos^2 3t + 25}$

$= \sqrt{36+25} = \sqrt{61}\ \dfrac{m}{s}$

dimension of velocity

$\approx 7.81\ \dfrac{m}{s}$

(b) Find the unit tangent vector at time t.

Soln: To get the unit vector in direction of $\vec{v} = a\,\underset{\sim}{i} + b\,\underset{\sim}{j} + c\,\underset{\sim}{k}$ divide by length:

$$\vec{T} = \frac{\vec{v}}{\|\vec{v}\|} = \frac{a\,\underset{\sim}{i} + b\,\underset{\sim}{j} + c\,\underset{\sim}{k}}{\sqrt{a^2+b^2+c^2}}$$

$$= \frac{a}{\sqrt{a^2+b^2+c^2}}\,\underset{\sim}{i} + \frac{b}{\sqrt{a^2+b^2+c^2}}\,\underset{\sim}{j} + \frac{c}{\sqrt{a^2+b^2+c^2}}\,\underset{\sim}{k}$$

Check: $\|\vec{T}\| = \left\| \frac{\vec{v}}{\|\vec{v}\|} \right\| = \frac{1}{\|\vec{v}\|}\|\vec{v}\| = 1$ ✓

Thus, we have helix

$$\vec{r}(t) = 2(\cos 3t\, \underset{\sim}{i} + \sin 3t\, \underset{\sim}{j}) + 5t\, \underset{\sim}{k}$$

with

$$\vec{v}(t) = -6\sin 3t\, \underset{\sim}{i} + 6\cos 3t\, \underset{\sim}{j} + 5\, \underset{\sim}{k}$$

$$\|\vec{v}\| = \sqrt{61}$$

So:

$$\vec{T}(t) = \frac{\vec{v}(t)}{\|\vec{v}(t)\|} = \frac{\vec{v}(t)}{\sqrt{61}}$$

$$= -\frac{6\sin 3t}{\sqrt{61}}\, \underset{\sim}{i} + \frac{6\cos 3t}{\sqrt{61}}\, \underset{\sim}{j} + \frac{5}{\sqrt{61}}\, \underset{\sim}{k}$$

$$= \overrightarrow{\left(-\frac{6\sin 3t}{\sqrt{61}}, \frac{6\cos 3t}{\sqrt{61}}, \frac{5}{\sqrt{61}}\right)} \checkmark$$

- **Ex** $\vec{r}(t) = \underbrace{\cos t}_{x(t)}\, \underset{\sim}{i} + \underbrace{t^2}_{y(t)}\, \underset{\sim}{j} + \underbrace{e^t}_{z(t)}\, \underset{\sim}{k}$

Find the speed $\dfrac{ds}{dt}$

Soln: $\vec{v}(t) = -\sin t\, \underset{\sim}{i} + 2t\, \underset{\sim}{j} + e^t\, \underset{\sim}{k}$

$$\frac{ds}{dt} = \|\vec{v}(t)\| = \sqrt{\sin^2 t + 4t^2 + e^{2t}}$$

2D

- Arclength –

Find the length of the helix

$$\vec{r}(t) = \cos t\, \underset{\sim}{i} + \sin t\, \underset{\sim}{j} + 8t\, \underset{\sim}{k}$$

between $t=1$ and $t=10$.

Soln: $\frac{ds}{dt} = \|\vec{v}(t)\|$ so $ds = \|\vec{v}(t)\| dt$

$$\vec{v}(t) = -\sin t\, \underset{\sim}{i} + \cos t\, \underset{\sim}{j} + 8\, \underset{\sim}{k}$$

$$\text{Length} = \int_{t=1}^{t=10} ds = \int_{1}^{10} \|\vec{v}(t)\| dt$$

$$\|\vec{v}(t)\| = \sqrt{\sin^2 t + \cos^2 t + 8} = \sqrt{9} = 3$$

$$\text{Length} = \int_1^{10} 3\, dt = 3t\Big]_1^{10} = 30 - 3 = \boxed{27}$$